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Curso: ECUACIONES DIFERENCIALES
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Estudiante 1.......................................
Estudiante 2.......................................

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Actividad #2

Harvesting a Renewable Resource. Suppose that the population y of a certain species of fish (for example, tuna or halibut) in a given area of the ocean is described by the logistic equation

$$dy/dt = r(1 - y/K)y.$$

While it is desirable to utilize this source of food, it is intuitively clear that if too many fish are caught, then the fish population may be reduced below a useful level, and possibly even driven to extinction. Problems 20 and 21 explore some of the questions involved in formulating a rational strategy for managing the fishery.[8]

20. At a given level of effort, it is reasonable to assume that the rate at which fish are caught depends on the population y: The more fish there are, the easier it is to catch them. Thus we assume that the rate at which fish are caught is given by Ey, where E is a positive constant, with units of 1/time, that measures the total effort made to harvest the given species of fish. To include this effect, the logistic equation is replaced by

$$dy/dt = r(1 - y/K)y - Ey. \qquad \text{(i)}$$

This equation is known as the **Schaefer model** after the biologist, M. B. Schaefer, who applied it to fish populations.
(a) Show that if $E < r$, then there are two equilibrium points, $y_1 = 0$ and $y_2 = K(1 - E/r) > 0$.
(b) Show that $y = y_1$ is unstable and $y = y_2$ is asymptotically stable.

(c) A sustainable yield Y of the fishery is a rate at which fish can be caught indefinitely. It is the product of the effort E and the asymptotically stable population y_2. Find Y as a function of the effort E; the graph of this function is known as the yield-effort curve.
(d) Determine E so as to maximize Y and thereby find the **maximum sustainable yield** Y_m.

21. In this problem we assume that fish are caught at a constant rate h independent of the size of the fish population. Then y satisfies

$$dy/dt = r(1 - y/K)y - h. \qquad \text{(i)}$$

The assumption of a constant catch rate h may be reasonable when y is large, but becomes less so when y is small.
(a) If $h < rK/4$, show that Eq. (i) has two equilibrium points y_1 and y_2 with $y_1 < y_2$; determine these points.
(b) Show that y_1 is unstable and y_2 is asymptotically stable.
(c) From a plot of $f(y)$ versus y show that if the initial population $y_0 > y_1$, then $y \to y_2$ as $t \to \infty$, but that if $y_0 < y_1$, then y decreases as t increases. Note that $y = 0$ is not an equilibrium point, so if $y_0 < y_1$, then extinction will be reached in a finite time.
(d) If $h > rK/4$, show that y decreases to zero as t increases regardless of the value of y_0.
(e) If $h = rK/4$, show that there is a single equilibrium point $y = K/2$ and that this point is semistable (see Problem 7). Thus the maximum sustainable yield is $h_m = rK/4$, corresponding to the equilibrium value $y = K/2$. Observe that h_m has the same value as Y_m in Problem 20(d). The fishery is considered to be overexploited if y is reduced to a level below $K/2$.

23. Some diseases (such as typhoid fever) are spread largely by *carriers*, individuals who can transmit the disease, but who exhibit no overt symptoms. Let x and y, respectively, denote

the proportion of susceptibles and carriers in the population. Suppose that carriers are identified and removed from the population at a rate β, so

$$dy/dt = -\beta y. \tag{i}$$

Suppose also that the disease spreads at a rate proportional to the product of x and y; thus

$$dx/dt = \alpha xy. \tag{ii}$$

(a) Determine y at any time t by solving Eq. (i) subject to the initial condition $y(0) = y_0$.
(b) Use the result of part (a) to find x at any time t by solving Eq. (ii) subject to the initial condition $x(0) = x_0$.
(c) Find the proportion of the population that escapes the epidemic by finding the limiting value of x as $t \to \infty$.

16. Another equation that has been used to model population growth is the Gompertz equation:

$$dy/dt = ry \ln(K/y),$$

where r and K are positive constants.
(a) Sketch the graph of $f(y)$ versus y, find the critical points, and determine whether each is asymptotically stable or unstable.
(b) For $0 \le y \le K$ determine where the graph of y versus t is concave up and where it is concave down.
(c) For each y in $0 < y \le K$ show that dy/dt as given by the Gompertz equation is never less than dy/dt as given by the logistic equation.

19. Consider a cylindrical water tank of constant cross section A. Water is pumped into the tank at a constant rate k and leaks out through a small hole of area a in the bottom of the tank. From Torricelli's theorem in hydrodynamics it follows that the rate at which water flows through the hole is $\alpha a\sqrt{2gh}$, where h is the current depth of water in the tank, g is the acceleration due to gravity, and α is a contraction coefficient that satisfies $0.5 \le \alpha \le 1.0$.
(a) Show that the depth of water in the tank at any time satisfies the equation

$$dh/dt = (k - \alpha a\sqrt{2gh})/A.$$

(b) Determine the equilibrium depth h_e of water and show that it is asymptotically stable. Observe that h_e does not depend on A.