UNIVERSIDAD AUTÓNOMA DE OCCIDENTE
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Curso: CÁLCULO II (C.E)
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método de sustitución y partes

Ej.: $\int \frac{\operatorname{sen}\sqrt{x}}{\sqrt{x}}\,dx$

$u = f(x)$
$du = f'(x).dx$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}.dx \Rightarrow 2du = \frac{dx}{\sqrt{x}}$

$\int \operatorname{sen} u \,.\, 2du = 2\int \operatorname{sen} u\, du = -2\cos u + C$

$= -2\cos\sqrt{x} + C$

una identidad importante

$\operatorname{sen} 2x = 2\operatorname{sen} x.\cos x$

Ej.: $\int \frac{\operatorname{sen}\sqrt{x}\cos\sqrt{x}}{\sqrt{x}}\,dx$

Sol. Método 1: $u = \sqrt{x}$

$2du = \frac{dx}{\sqrt{x}}$

$\int \frac{\operatorname{sen}\sqrt{x}\cos\sqrt{x}}{\sqrt{x}}\,dx =$

$= \int \operatorname{sen} u.\cos u \,.\, 2du =$

$= 2.\int \operatorname{sen} u.\cos u\, du = ?$

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$

$z = \operatorname{sen} u$

$dz = \cos u.\,du$

Entonces:

$\int \frac{\operatorname{sen}\sqrt{x}\cos\sqrt{x}}{\sqrt{x}}\,dx = 2\int z.\,dz = \cancel{2}.\frac{z^2}{\cancel{2}} + C = z^2 + C$

$= \operatorname{sen}^2 u + C = \operatorname{sen}^2\sqrt{x} + C$

Método 2 :

$$\int \frac{\operatorname{sen}\sqrt{x}\cdot\cos\sqrt{x}}{\sqrt{x}}\,dx$$

s/.

$u = \operatorname{sen}\sqrt{x}$

$du = \cos\sqrt{x}\cdot\frac{1}{2\sqrt{x}}\cdot dx$

$2\,du = \frac{\cos\sqrt{x}}{\sqrt{x}}\cdot dx$

$$\int \frac{\operatorname{sen}\sqrt{x}\cdot\cos\sqrt{x}}{\sqrt{x}}\,dx = 2\int u\,du = \cancel{2}\cdot\frac{u^2}{\cancel{2}} = u^2+C$$

$$= \operatorname{sen}^2\sqrt{x}+C$$

1: $\int x\sqrt{3x+1}\,dx$

s/.

$u = 3x+1$

$du = 3\cdot dx$

$\frac{u-1}{3} = x$

$$\int x\sqrt{3x+1}\,dx = \int \frac{u-1}{3}\cdot\sqrt{u}\,\frac{du}{3}$$

$$= \frac{1}{9}\int (u-1)\cdot\sqrt{u}\,du =$$

$$= \frac{1}{9}\int (u^{3/2}-u^{1/2})\,du =$$

$$= \frac{1}{9}\cdot\left(\frac{u^{5/2}}{5/2}-\frac{u^{3/2}}{3/2}\right)+C = \frac{1}{9}\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)+C$$

$$= \frac{1}{9}\left(\frac{2}{5}(3x+1)^{5/2}-\frac{2}{3}(3x+1)^{3/2}\right)+C$$

Integración por partes

$$(f(x)\cdot g(x))' = f'(x)g(x) + f(x)\cdot g'(x) \Rightarrow$$

$$\int (f(x)\cdot g(x))'\,dx = f(x)g(x) = \int \left(f'(x)g(x) + f(x)g'(x)\right)dx$$

$$\Rightarrow \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx = f(x)\cdot g(x) \quad (*)$$

Haciendo el sgte. cambio de vble.:

$$\begin{array}{ll} u = f(x) & dv = g'(x)\,dx \\ du = f'(x)\,dx & v = g(x) \end{array}$$

deriv. intgr.

Reescribiendo (*):

$$\int v\,du + \int u\cdot dv = u\cdot v \Rightarrow$$

$$\int u\,dv = u\cdot v - \int v\cdot du$$

Fórmula de integración por partes

Ej: Calcule $\int \underbrace{x e^{x^2} dx}_{*}$

S/ No es necesario aplicar Int. por partes.

Usando sustitución:

$u = e^{x^2}$

$du = 2\underbrace{x . e^{x^2} dx}_{*}$

$$\int x e^{x^2} dx = \int \frac{1}{2} du = \frac{1}{2} u + c = \frac{1}{2} \cdot e^{x^2} + c$$

Ej: Calcule $\int x e^x dx$

S/ Por partes:

$u = e^x \qquad dv = x\, dx$

$du = e^x dx \qquad v = \frac{x^2}{2}$

$$\int x e^x dx = \frac{x^2}{2} e^x - \underbrace{\int \frac{x^2}{2} \cdot e^x dx}_{}$$

$\Rightarrow$ indica una mala elección de u

Otro "ensayo":

$u = x \qquad dv = e^x dx$

$du = dx \qquad v = e^x$

$$\int x e^x dx = x e^x - \int e^x . dx = x e^x - e^x + c =$$

$$e^x (x - 1) + c$$

Para mi:

1. $\int x \ln x \, dx$

2. $\int x^2(3x^2+5)\, dx$

3. $\int x \operatorname{sen} x$

4. $\int \ln x \, dx$

5. $\int x^2 \ln x \, dx$

6. $\int x^2 \cos x \, dx$

7. $\int e^x \operatorname{sen} x \, dx$

8. $\int x\sqrt{3x+1}\, dx$

9. $\int e^{\sqrt{x}}\, dx$

10. $\int x \operatorname{sen}^{-1} x \, dx$

Ej: Calcule la función posición $x(t)$ de un móvil, sabiendo que su velocidad es $v(t) = \operatorname{sen} t \cos t$ y $x(\frac{\pi}{2}) = 1$

S/.

$$x(t) = \int v(t)\,dt = \int \operatorname{sen} t . \cos t \; dt$$

$$u = \operatorname{sen} t$$
$$du = \cos t . dt$$

REM.

Si $y = f(x)$, el diferencial de y es:
$$dy = f'(x).dx$$

Entonces: $\int \operatorname{sen} t . \cos t \, dt = \int u . du = \frac{1}{2}u^2 + C = \frac{1}{2}\operatorname{sen}^2 t + C$

$x(t) = \frac{1}{2}\operatorname{sen}^2 t + C$. Dado que $x(\frac{\pi}{2}) = 1$, entonces:

$x(\pi/2) = \frac{1}{2}\operatorname{sen}^2(\frac{\pi}{2}) + C = 1 \Rightarrow C = 1 - \frac{1}{2}\operatorname{sen}^2(\pi/2) \Rightarrow$

$C = 1 - \frac{1}{2} \Rightarrow C = \frac{1}{2}$. En resumen:

$$\boxed{x(t) = \frac{1}{2}\operatorname{sen}^2 t + \frac{1}{2}}$$

En el ejemplo anterior se ha utilizado el método de integración por sustitución. Este método es útil a la hora de resolver integrales de la forma $\int f(g(x))g'(x)\,dx$, haciendo una sustitución. Veamos:

$$\text{Sea}\quad \begin{array}{l} u = g(x) \\ du = g'(x)\cdot dx \end{array} \Rightarrow \int f(g(x))\,g'(x)\,dx = \underbrace{\int f(u)\cdot du}_{\text{Se espera que sea más sencilla !!}}$$

Ej: Calcule $\int 2x\sqrt{x^2+5}\,dx$

S/. $\begin{array}{l} u = x^2+5 \\ du = 2x\,dx \end{array} \Rightarrow \int 2x\sqrt{x^2+5}\,dx = \int \sqrt{u}\cdot du =$

$$= \frac{2}{3}u^{3/2} + C = \frac{2}{3}(x^2+5)^{3/2} + C$$

ξ:: Calculate $\int (y+1)^4 dy \cdot \int \frac{1}{(y-10)^7} dy$

S/.

$$\int (y+1)^4 dy \qquad u = y+1$$
$$du = dy$$

$$\int (y+1)^4 dy = \int u^4 du = \frac{1}{5} u^5 + C_1 = \frac{1}{5}(y+1)^5 + C_1$$

— o —

$$\int \frac{1}{(y-10)^7} dy \qquad u = y-10$$
$$du = dy$$

$$\int \frac{1}{(y-10)^7} dy = \int \frac{1}{u^7} du = \int u^{-7} du = -\frac{1}{6} u^{-6} + C_2$$

$$= -\frac{1}{6} \cdot \frac{1}{(y-10)^6} + C_2$$

$$\int (y+1)^4 dy \cdot \int \frac{1}{(y-10)^7} dy = \left(\frac{1}{5}(y+1)^5 + C_1\right)\left(-\frac{1}{6} \frac{1}{(y-10)^6} + C_2\right)$$

Algunas identidades útiles :

$$\operatorname{sen}^2 x + \cos^2 x = 1 \quad , \quad 1+\cot^2 x = \csc^2 x \quad , \quad 1+\tan^2 x = \sec^2 x$$

$$\operatorname{sen}^2 x = \frac{1-\cos 2x}{2} \quad , \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

$$\operatorname{sen}(2x) = 2 \operatorname{sen} x . \cos x \quad , \quad \cos(2x) = \cos^2 x - \operatorname{sen}^2 x$$

Ej: Calcule $\int \cos(2x)\,dx$

s/.

$$u = 2x$$
$$du = 2.dx \Rightarrow \frac{du}{2} = dx$$

$$\int \cos(2x)\,dx = \int \cos u . \frac{du}{2} = \frac{1}{2}.\int \cos u\, du =$$

$$= \frac{1}{2}.\operatorname{sen} u + C = \frac{1}{2}\operatorname{sen}(2x) + C$$

Ej: Calcule $\int \operatorname{sen}^2 x\,dx$

s/

$$\int \operatorname{sen}^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx$$

$$= \frac{1}{2}\left(x - \frac{1}{2}\operatorname{sen}(2x)\right) + C$$

Ej: Calcule $\int x\sqrt{x+1}\,dx$

S/.
$u = x+1 \Rightarrow x = u-1$
$du = dx$

$$\int x\sqrt{x+1}\,dx = \int (u-1)\sqrt{u}\,du = \int (u^{3/2} - u^{1/2})\,du$$

$$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$$

Ej: Calcule $\int \frac{\operatorname{sen}^2\sqrt{x}\cos\sqrt{x}}{\sqrt{x}}\,dx$

S/.
$u = \sqrt{x}$ $\qquad 2\int \operatorname{sen}^2 u \cdot \cos u\,du$
$du = \frac{1}{2\sqrt{x}}dx$

$2du = \frac{1}{\sqrt{x}}dx$

Sustituyendo nuevamente: $v = \operatorname{sen} u$
$dv = \cos u\,du$

$$\int \frac{\operatorname{sen}^2\sqrt{x}\cos\sqrt{x}}{\sqrt{x}}\,dx = 2\int \operatorname{sen}^2 u\cos u\,du = 2\int v^2\cdot dv =$$

$$= \frac{2}{3}v^3 + C = \frac{2}{3}\cdot\operatorname{sen}^3 u + C = \frac{2}{3}\operatorname{sen}^3(\sqrt{x}) + C$$

Integración por partes

$$(f(x).g(x))' = f'(x)g(x) + f(x)g'(x) \quad \Rightarrow$$

$$f(x)g(x) = \int \left(f'(x)g(x) + f(x)g'(x)\right)dx = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

Usando el método de sustitución

derivar

$u = f(x)$ $\qquad\qquad dv = g'(x)dx$

integrar

$du = f'(x)dx$ $\qquad\qquad v = g(x)$

Entonces:

$$u.v = \int v\,du + \int u.\,dv \quad \Rightarrow$$

$$\underbrace{\int u\,dv = u.v - \int v\,du}_{\text{Fórmula de integración por partes}}$$

$$\int xe^x dx \qquad u = x \qquad dv = e^x dx$$

$$du = dx \qquad v = e^x$$

$$\int xe^x dx = x.e^x - \int e^x dx = xe^x - e^x + C$$

[redacted]: 1a. $\int \frac{x^3+x-1}{x^2}\,dx = \int \left(x + \frac{1}{x} - x^{-2}\right) dx =$

$= \frac{x^2}{2} + \ln|x| + \frac{1}{x} + C$

1b. $\int \frac{\ln\sqrt{x}}{x}\,dx$

$u = \ln\sqrt{x}$

$du = \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}\,dx$

$du = \frac{1}{2x}\,dx$

$2du = \frac{1}{x}\,dx$

$\int 2u\,du = u^2 + c = (\ln\sqrt{x})^2 + C$

1c. $\int \frac{x}{\sqrt{2x^2+2}}\,dx$

$u = 2x^2+2$

$du = 4x\,dx$

$\frac{du}{4} = x\,dx$

$\frac{1}{4}\int \frac{du}{\sqrt{u}} = \frac{1}{4}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{4}\int u^{-\frac{1}{2}}\,du =$

$= \frac{1}{4}\,\frac{u^{1/2}}{1/2} + C = \frac{1}{2}\sqrt{2x^2+2} + C$

2. $v(t) = \frac{\ln^2 t}{t}$ $\quad x(0) = ?$ $\quad x(1) = 2$

$$x(t) = \int \frac{\ln^2 t}{t}\,dt \qquad u = \ln t$$
$$du = \frac{1}{t}\,dt$$

$$x(t) = \int u^2\,du = \frac{1}{3}u^3 + c = \frac{1}{3}(\ln t)^3 + c$$

$$x(1) = 2 \Rightarrow x(1) = \cancel{\frac{1}{3}(\ln 1)^3} + c = 2 \Rightarrow \boxed{c = 2}$$

$$\boxed{x(t) = \frac{1}{3}(\ln t)^3 + 2} \Rightarrow x(0) \text{ no está def.}$$

3. ¿ $\int \ln x\,dx = x\ln x + c$?

$$(x\ln x + c)' = \ln x + x \cdot \frac{1}{x} + 0 = \boxed{\ln x + 1}$$

R/. NO!!

3. $\int x \ln x\,dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$

$$\left(\frac{1}{2}x^2 \cdot \ln x - \frac{1}{4}x^2 + c\right)' = x\ln x + \cancel{\frac{1}{2}x^2 \cdot \frac{1}{x}} - \cancel{\frac{1}{2}x} + 0$$
$$= x\ln x$$

Integración por partes:

Recuerde que $\int u\,dv = u.v - \int v\,du$

Ej: $\int x \ln x\,dx$

$$\begin{cases} u = x & dv = \ln x\,dx \\ du = dx & v = ?? \end{cases}$$

(Mala elección de u)

$$\begin{matrix} u = \ln x & dv = x\,dx \\ du = \frac{1}{x}dx & v = \frac{x^2}{2} \end{matrix}$$

$$\int x\ln x\,dx = \frac{x^2}{2}.\ln x - \int \frac{x^2}{2}\cdot\frac{1}{x}dx = \frac{x^2}{2}.\ln x - \int \frac{x}{2}dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}.\frac{x^2}{2} + c = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$$

Ej: Calcule: 1. $\int xe^x dx$ 2. $\int \ln x\,dx$ 3. $\int e^{\sqrt{x}}dx$

4. $\int (x+1)\sqrt{x+1}\,dx$ 5. $\int \sec x\,dx$

S/. 1. $\int xe^x dx$

$$\begin{matrix} u = x & dv = e^x dx \\ du = dx & v = e^x \end{matrix}$$

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C =$$
$$= e^x(x-1) + C$$

2. $$\begin{matrix} u = \ln x & dv = dx \\ du = \frac{1}{x}dx & v = x \end{matrix}$$

$$\int \ln x\,dx = x\ln x - \int x.\frac{1}{x}dx$$
$$= x\ln x - \int 1\,dx = x\ln x - x + c$$

3. $\int e^{\sqrt{x}}dx$

S/. Inicialmente por sustitución: $w=\sqrt{x}$

$dw=\frac{1}{2\sqrt{x}}dx$

$2\sqrt{x}\,dw=dx$

$2w\,dw=dx$

$\int e^{\sqrt{x}}dx=\int e^{w}\cdot 2w\,dw=$

$=2\int we^{w}dw$.

Ahora por partes:

$u=w \qquad dv=e^{w}dw$
$du=dw \qquad v=e^{w}$

$\int we^{w}dw=we^{w}-\int e^{w}dw$
$=we^{w}-e^{w}+C$

Finalmente: $\int e^{\sqrt{x}}dx=2(we^{w}-e^{w})+C$
$=2(\sqrt{x}e^{\sqrt{x}}-e^{\sqrt{x}})+C$

4. $\int (x+1)\sqrt{x+1}\,dx$

S/. Por sustitución $u=x+1$, $du=dx$

$\int (x+1)\sqrt{x+1}\,dx=\int u\sqrt{u}\,du=$
$=\int u^{3/2}du=\frac{2}{5}u^{5/2}+C=\frac{2}{5}(x+1)^{5/2}+C$

Por partes: $u=x+1 \qquad dv=\sqrt{x+1}\,dx$
$du=dx \qquad v=\frac{2}{3}(x+1)^{3/2}$

$\int (x+1)\sqrt{x+1}\,dx=\frac{2}{3}(x+1)^{5/2}-\frac{2}{3}\int (x+1)^{3/2}dx$
$=\frac{2}{3}(x+1)^{5/2}-\frac{2}{3}\cdot\frac{2}{5}(x+1)^{5/2}+C=$
$=\frac{2}{5}(x+1)^{5/2}+C$

$u=\sqrt{x+1}$
$du=\frac{1}{2\sqrt{x+1}}dx$

$2u\,du=dx$

$\int u^2\cdot u\,2u\,du=$
$2\int u^4du=$
$\frac{2}{5}u^5+C=$
$\frac{2}{5}(x+1)^{5/2}+C$