Differential Equations (ENG.)

Classification of Differential Equations

Víctor Hugo Gil Avendaño[1]

[1]Dpto. de Matemáticas

Universidad Autónoma de Occidente

vhgil@uao.edu.co

DifferentialEquations

Recall that a differential equation is an equation (has an equal sign) that involves derivatives. Just as biologists have a classification system for life, mathematicians have a classification system for differential equations. We can place all differential equation into two types: ordinary differential equation and partial differential equations.

1. Partial vs. Ordinary Differential Equations

An **ordinary differential equation** is an equation relating a function and its derivatives with respect to a single variable.

A **partial differential equation** is one relating a function of more than one variable to its partial derivatives.

Example 1.

$$\frac{d\phi(x)}{dx} + g(x)\,\phi(x) = 0$$

is an ordinary differential equation, while

$$\frac{\partial f(x,y)}{\partial x} + G(x,y)\,f(x,y) = 0$$

is a partial differential equation.

In this course we shall deal almost exclusively with ordinary differential equations.

2. The order of a Differential Equations

The **order** of a differential equation is the highest number of derivatives appearing in the equation. Thus,

$$2x^2 \frac{df}{dx} + \sqrt{x}\, \frac{d^3 f}{dx^3} + e^x = 2$$

is an third order, ordinary, differential equation, while

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{\partial^2 \phi}{\partial t^2} = 0$$

is a second order, partial, differential equation.

Example 2.

- $$\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^4 = 3x \sin x$$

  is a second order differential equation, since a second derivative appears in the equation.

- $$3y^4\, y''' - x^3\, y' + e^{xy} y = 0$$

  is a third order differential equation.

3. Linear vs Non-Linear Differential Equations

An ordinary or partial differential equation is said to be **linear** if it is linear in the "unknowns" $y, y', y'', ..., y^{(n)}$. Thus, a general, linear, ordinary, n^{th} order, differential equation would be one of the form

$$a_n(x)\,\frac{d^n y}{dx^n} + a_{n-1}(x)\,\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1(x)\,\frac{dy}{dx} + y = b(x)$$

It is important to note that the functions $a_n(x), a_{n-1}(x), ..., a_1(x), g(x)$ need not be linear functions of x. The following two examples should convey the general idea.

Example 3.

- $$x^2\,\frac{\partial f}{\partial x} + z\,\frac{\partial^2 f}{\partial y^2} = e^{xyz}$$

  is a 2^{nd} order, linear, partial, differential equation.

- $$\frac{d^3 y}{dx^3} + x^2\,\frac{dy}{x}y^2 = 1$$

  is a non-linear, ordinary, differential equation of order 3. The equation is non-linear arises because of the presence of the term y^2 which is a quadratic function of the unknown function y.

Stop!

$$\left(\frac{dy}{dx}\right)^2 \neq \frac{d^2y}{dx^2}$$

4. Homogeneous vs. heterogeneous

A differential equation is **homogeneous** if it contains no non-differential terms and **heterogeneous** if it does.

Example 4.

$$\frac{dy}{dx} = \mathbf{ax}$$

and

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = \mathbf{b}$$

are heterogeneous (unless the coefficients a and b are zero), but

$$\frac{dy}{dx} + y\cos x = 0$$

is homogeneous.

A zero right-hand side is a sign of a tidied-up homogeneous differential equation, but beware of non-differential terms hidden on the left-hand side!