Calculus II (ENG.)

Functions of Several Variables, Graphs, Limits and continuity

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Functions of Several Variables

Tools to learn for functions of several variables

- Evaluating functions.
- Finding domain and range.
- Contour Maps (functions of 2 variables)
- Using functions of several variables to model real-life situations (how to ask and answer questions).

1. Definition of function of 2 variables

A **function of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a subset D of the plane a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) : (x, y) \in D\}$.

We often write $z = f(x, y)$ to make explicit the values taken on by f at the general point (x, y). The variables x and y are **independent variables** and z is the **dependent variable**.

Functions of more than two variables can be defined similarly.

For example: $f(x, y) = x^3 - 3y^2$ is a function which gives an output for every pair of input values.
$f(0, 0) = (0)^3 - 3(0)^2 = 0$

$f(1,2) = (1)^3 - 3(2)^2 = -11$
$f(-4,3) = (-4)^3 - 3(3)^2 = 37$

The domain of a function is very important and can be either

- **Specified by the problem, i.e. specific restrictions are given**

$$f(x,y) = x^2 + y^2 \text{ such that } -1 < x < 1, \ -1 < y < 1$$

- **Assumed to be all points for which the function is valid**

$$f(x,y) = \sqrt{4 - x^2 - y^2}$$

Recall from functions of one variable

- **You cannot take the square root of a negative number**, so if $f(x,y) = \sqrt{2x+3y}$ the assumed domain requires that $2x + 3y \geq 0$.
- **You cannot divide by zero**, so if $f(x,y,z) = \frac{1}{x-y^2+3z}$ the assumes domain requires that $x - y^2 + 3z \neq 0$.
- **You cannot take the logarithm of 0 or a negative number**, so if $f(x,y) = Ln(x^2 - 3y)$ the assumed domain requires that $x^2 - 3y > 0$.

Example 1. Let

$$f(x,y) = \frac{\sqrt{x+y+1}}{x-1}$$

- Evaluate $f(3,2)$, $f(2,-5)$, $f(1,-1)$, and $f(-1,4)$.
- Find the domain of f.

Example 2. Determine the domain of each of the following.

a) $f(x,y) = \sqrt{x+y}$

b) $f(x,y) = \sqrt{x} + \sqrt{y}$

c) $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2-16}}$

d) $f(x,y) = xLn(y^2 - x)$

e) $f(x,y) = \sqrt{xy}$

2. Graphs

If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x,y,z) in R^3 such that $z = f(x,y)$ and $(x,y) \in D$.

$$Graph(f) = \{(x,y,z) \in R^3 / z = f(x,y),\ (x,y) \in D\}$$

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The next topic that we should look at is that of **level curves** or **contour curves**. The level curves of the function f are two dimensional curves we get by setting $z = c$, where c is any number. So the equations of the level curves are $f(x, y) = c$. Note that sometimes the equation will be in the form $f(x, y, z) = 0$ and in these cases the equations of the level curves are $f(x, y, k) = 0$.

You've probably seen level curves (or contour curves, whatever you want to call them) before. If you've ever seen the elevation map for a piece of land, this is nothing more than the contour curves for the function that gives the elevation of the land in that area. Of course, we probably don't have the function that gives the elevation, but we can at least graph the contour curves. Let's do a quick example of this.

Example 3. Identify the level curves of each of the following. Sketch a few of them

a) $f(x, y) = \sqrt{x^2 + y^2}$

b) $f(x, y) = x^2 - y^2$

c) $f(x, y) = x^2 + y^2$

d) $f(x, y) = 16 - 4x^2 - y^2$

e) $f(x, y) = \sqrt{4 - x^2 - y^2}$

Exercises 1.

a) From section 13.1 of the guide text Calculus 9th Edition by Ron Larson, Bruce H. Edwards, perform the exercises 76, 77, 78, 82, 83.

3. Limits

In this section we will take a look at limits involving functions of more than one variable. In fact, we will concentrate mostly on limits of functions of two variables, but the ideas can be extended out to functions with more than two variables.

Before getting into this let's briefly recall how limits of functions of one variable work. We say that,

$$\lim_{x \to a} f(x) = L$$

provided,

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

In other words, we will have $\lim_{x \to a} f(x) = L$ provided $f(x)$ approaches L as we move in towards $x = a$ (without letting $x = a$) from both sides.

Now, notice that in this case there are only two paths that we can take as we move in towards $x = a$. We can either move in from the left or we can move in from the right. Then in order for the limit of a function of one variable to exist the function must be approaching the same value as we take each of these paths in towards $x = a$.

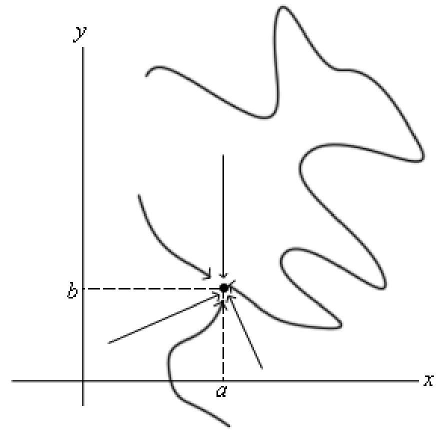
With functions of two variables we will have to do something similar, except this time there is (potentially) going to be a lot more work involved. Let's first address the notation and get a feel for just what we're going to be asking for in these kinds of limits.

We will be asking to take the limit of the function $f(x, y)$ as x approaches a and as y approaches b. This can be written in several ways. Here are a of the standard notation.

$$\operatorname{lím}_{(x,y)\to(a,b)} f(x, y)$$

This notation is also a little more helpful in illustrating what we are really doing here when we are taking a limit. In taking a limit of a function of two variables we are really asking what the value of $f(x, y)$ is doing as we move the point (x, y) in closer and closer to the point (a, b) without actually letting it be.

Just like with limits of functions of one variable, in order for this limit to exist, the function must be approaching the same value regardless of the path that we take as we move in towards (a, b). The problem that we are immediately faced with is that there are literally an infinite number of paths that we can take as we move in towards (a, b). Here are a few examples of paths that we could take.



We put in a couple of straight line paths as well as a couple of "stranger" paths that aren't straight line paths. Also, we only included 6 paths here and as you can see simply by varying the slope of the straight line paths there are an infinite number of these and then we would need to consider paths that aren't straight line paths.

In other words, to show that a limit exists we would technically need to check an infinite number of paths and verify that the function is approaching the same value regardless of the path we are using to approach the point.

Luckily for us however we can use one of the main ideas from Calculus I limits to help us take limits here.

Example 4. Determine if the following limits exist or not. If they do exist give the value of the limit.

a) $\lim\limits_{(x,y,z)\to(2,1,-1)} 3x^2z + yxcos(\pi x - \pi z)$

b) $\lim\limits_{(x,y)\to(5,1)} \dfrac{xy}{x+y}$

c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{x^4+3y^4}$

d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y}{x^6+y^2}$

4. Continuity

A function $f(x,y)$ is **Continuous** at the point if,

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$