UNIVERSIDAD DEL VALLE
FACULTAD DE CIENCIAS NATURALES Y EXACTAS
DEPARTAMENTO DE MATEMÁTICAS

Curso: CÁLCULO I
Profesor: Victor Hugo Gil A. *febrero*/2019

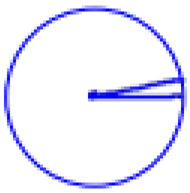
Funciones trigonométricas

TRIGONOMETRIA

Ángulos: ∢ abertura entre dos rectas que se intersectan

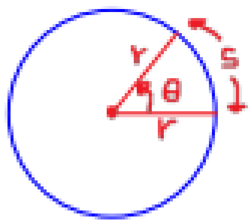
Medición: Sexagesimal (Grados) deg

$1^\circ = \frac{1}{360}$ de revolución

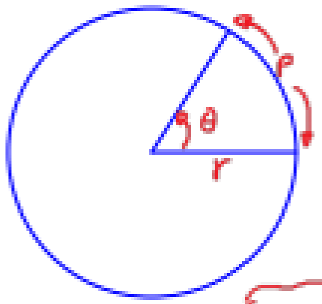


Una revolución mide 360°

Simétrico (Radianes) Rad



Se dice que la medida del ángulo θ es de un radián si $\frac{s}{r} = 1$



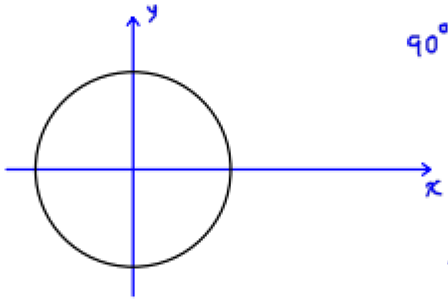
$m(\theta) = \frac{p}{r}$ rad

FUNDAMENTAL

$360^\circ = 2\pi$ rad $\Rightarrow$ $1^\circ = \frac{\pi}{180}$ rad

$\Rightarrow$ 1 rad $= \left(\frac{180}{\pi}\right)^\circ$

$\pi \simeq 3.1416$



$90^\circ = \frac{\pi}{2}$ rad

$180^\circ = \pi$ rad

$270^\circ = \frac{3\pi}{2}$ rad

$\frac{\pi}{6}$ rad $= (?)^\circ$

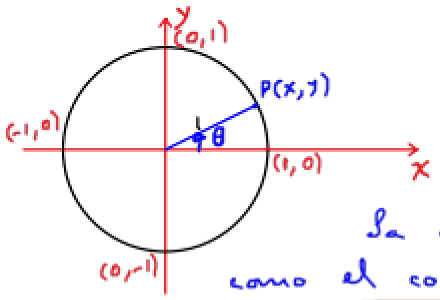
$1 \text{ rad} = \left(\frac{180}{\pi}\right)^\circ \Rightarrow \frac{\pi}{6} \text{ rad} = \left(\frac{180}{\cancel{\pi}} \cdot \frac{\cancel{\pi}}{6}\right)^\circ$

$\frac{\pi}{6}$ rad $= 30^\circ$

$45^\circ = (?)$rad $\Rightarrow 1^\circ = \frac{\pi}{180}$ rad $\Rightarrow 45^\circ = 45 \cdot \frac{\pi}{180}$ rad

$\Rightarrow 45^\circ = \frac{\pi}{4}$ rad

FUNCIONES CIRCULARES



El lado terminal del ángulo θ intersecta la circunferencia de radio 1, en un punto P(x, y)

La coordenada x se define como el coseno de θ y se denota así: $\boxed{x = \cos\theta}$

$y = \sin\theta$

La coordenada y se define como el seno de θ y se denota así: $\boxed{y = \operatorname{sen}\theta}$

De acuerdo con lo anterior:

$\operatorname{sen} \pi = 0 \qquad \cos \frac{\pi}{2} = 0 \qquad \cos \pi = -1 \quad \ldots$

Puede mostrarse que:

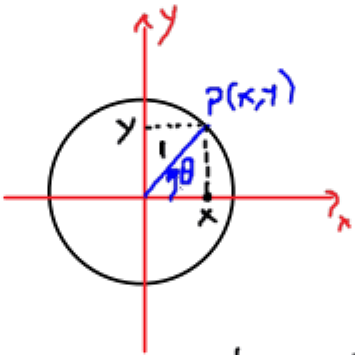
$$\operatorname{sen} \frac{\pi}{4} = \frac{\sqrt{2}}{2} \qquad \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

$$\operatorname{sen} \frac{\pi}{3} = \frac{\sqrt{3}}{2} \qquad \cos \frac{\pi}{3} = \frac{1}{2} \qquad \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \operatorname{sen} \frac{\pi}{6} = \frac{1}{2}$$

Se definen: $\tan\theta = \frac{\operatorname{sen}\theta}{\cos\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\operatorname{sen}\theta}$

$\cot\theta = \frac{\cos\theta}{\operatorname{sen}\theta}$

Propiedades:



$x^2 + y^2 = 1 \iff$

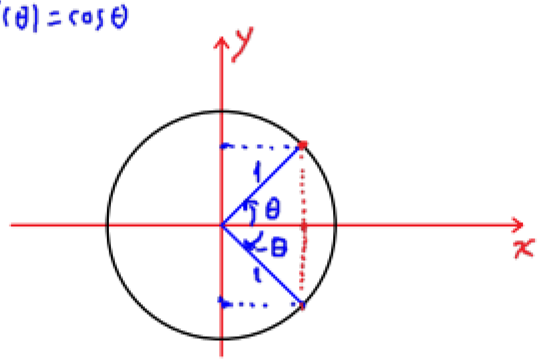
$\cos^2\theta + \operatorname{sen}^2\theta = 1$

$-1 \leq \operatorname{sen}\theta \leq 1 \iff |\operatorname{sen}\theta| \leq 1$

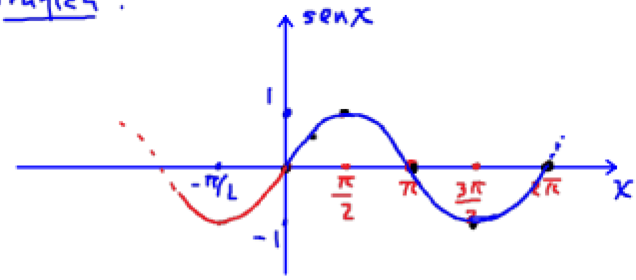
$-1 \leq \cos\theta \leq 1 \iff |\cos\theta| \leq 1$

$\cos\theta = \cos(-\theta) \Rightarrow f(\theta) = \cos\theta$
es una función par

$\operatorname{sen}(-\theta) = -\operatorname{sen}\theta \Rightarrow$
$g(\theta) = \operatorname{sen}\theta$ es una función impar



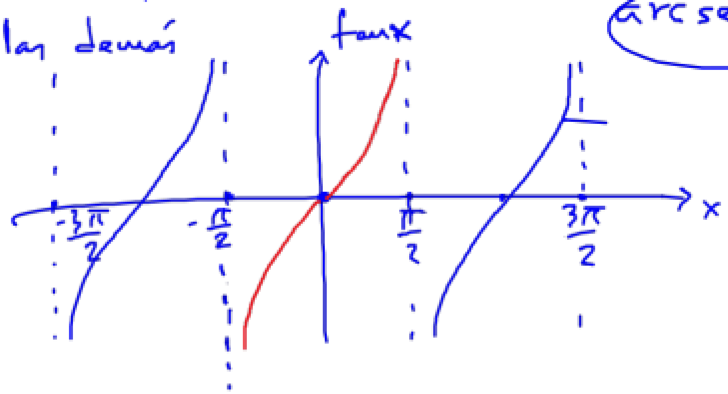
Gráfica:



x	0	π/2	π	3π/2	2π
sen x	0	1	0	-1	0

$$\frac{1}{\cos x}$$

Estudiar las demás



$\operatorname{arc\,sen} -1 = -\frac{\pi}{2}$

Algunas identidades útil.

$$\cos^2 x + \operatorname{sen}^2 x = 1 \quad \begin{cases} \operatorname{sen}^2 x = 1 - \cos^2 x \\ \cos^2 x = 1 - \operatorname{sen}^2 x \end{cases} \qquad (0)$$

Si en (*) divido por $\cos^2 x$:

$$1 + \tan^2 x = \sec^2 x \quad \begin{cases} \tan^2 x = \sec^2 x - 1 \\ 1 = \sec^2 x - \tan^2 x \end{cases}$$

Si en (*) divido por $\operatorname{sen}^2 x$:

$$\cot^2 x + 1 = \csc^2 x \quad \begin{cases} \cot^2 x = \csc^2 x - 1 \\ 1 = \csc^2 x - \cot^2 x \end{cases}$$

$$\operatorname{sen}(x \pm y) = \operatorname{sen} x \cdot \cos y \pm \cos x \cdot \operatorname{sen} y \qquad (1)$$

$$\cos(x \pm y) = \cos x \cos y \mp \operatorname{sen} x \cdot \operatorname{sen} y \qquad (2)$$

De (1) y (2) se obtienen:

$$\operatorname{sen}(2x) = 2 \operatorname{sen} x \cos x \qquad (3)$$

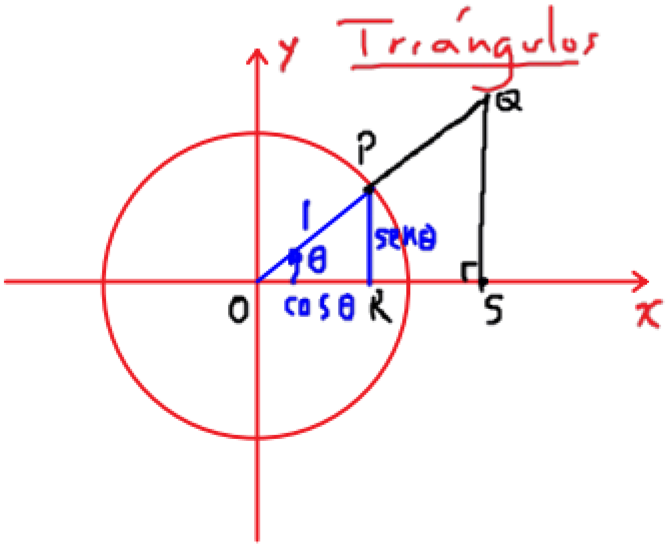
$$\cos(2x) = \cos^2 x - \operatorname{sen}^2 x \qquad (4) \text{ y } (0)$$

De (4): $\operatorname{sen}^2 x = \frac{1 - \cos 2x}{2}$ $\quad \cos^2 x = \frac{1 + \cos 2x}{2}$

D/

$$\cos(2x) = \cos^2 x - \operatorname{sen}^2 x = 1 - \operatorname{sen}^2 x - \operatorname{sen}^2 x = 1 - 2\operatorname{sen}^2 x$$

$$\operatorname{sen}^2 x = \frac{1 - \cos(2x)}{2}$$



$\triangle OPR \sim \triangle OQS$

$$\frac{|\overline{OP}|}{|\overline{OQ}|} = \frac{|\overline{OR}|}{|\overline{OS}|} \iff$$

$$\frac{1}{|\overline{OQ}|} = \frac{\cos\theta}{|\overline{OS}|} \implies$$

$$\cos\theta = \frac{|\overline{OS}|}{|\overline{OQ}|} = \frac{\text{cat. ady}}{\text{hip.}}$$

Con otra proporción muestre ud. que

$$\operatorname{sen}\theta = \frac{|\overline{QS}|}{|\overline{OQ}|} = \frac{\text{cat. op.}}{\text{hip.}}$$

Recordar: Si $y = f(x)$ y f es invertible, entonces $f^{-1}(y) = f^{-1}(f(x)) \iff f^{-1}(y) = x$. Entonces para funciones trigonométricas inversas:

$$\operatorname{sen}\theta = w \qquad w \in [-1, 1]$$

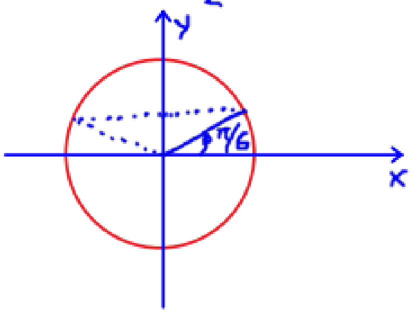
$$\theta = \operatorname{sen}^{-1}(w) = \operatorname{arcsen}(w)$$

Ecuaciones trigonométricas:

Resuelva la ecuación $2\operatorname{sen}\theta = 1$

1) Para $\theta \in \mathbb{R}$ 2) Para $0 \leq \theta \leq 2\pi$

S/. 1) $\operatorname{sen}\theta = \frac{1}{2} \implies \theta = \operatorname{sen}^{-1}\left(\frac{1}{2}\right)$



$$S = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}, \dots$$

$$= \left\{\frac{\pi}{6} + 2n\pi, n \in \mathbb{Z}^+ \cup \{0\}\right\} \cup \left\{\frac{5\pi}{6} + 2n\pi, n \in \mathbb{N}\right\}$$

2) $S = \left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$

Ej: Resuelva la ecuación

$$2\cos^2\theta + \cos\theta - 1 = 0 \qquad \theta \in [0, 2\pi]$$

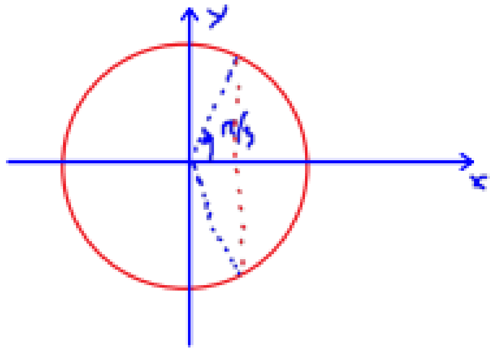
S/. Sea $\cos\theta = x$

$$2x^2 + x - 1 = 0 \iff 4x^2 + 1(2x) - 2 = 0$$

$$(2x+2)(2x-1) = 0 \iff 2(x+1)(2x-1) = 0 \iff$$

$$x+1=0 \quad \vee \quad 2x-1=0 \iff x=-1 \quad \vee \quad x=\frac{1}{2}$$

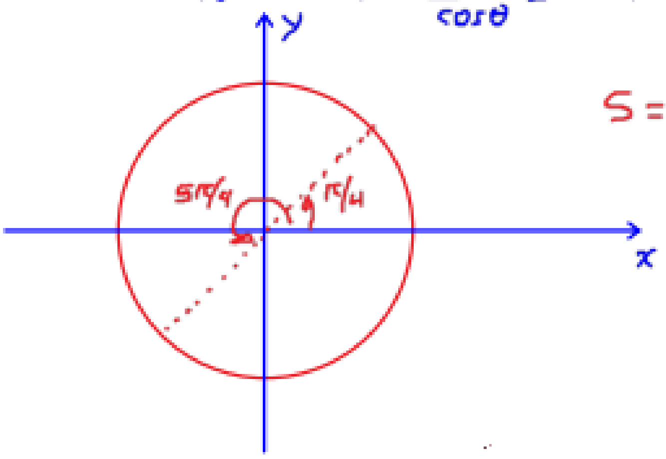
$$\iff \cos\theta = -1 \quad \vee \quad \cos\theta = \frac{1}{2} \iff \theta = \cos^{-1}(-1) \ \vee \ \theta = \cos^{-1}\left(\frac{1}{2}\right)$$



$$S = \left\{ \pi, \frac{\pi}{3}, \frac{5\pi}{3} \right\}$$

Ej: Resuelva la ecuación $\tan\theta = 1$, $\theta \in \mathbb{R}$

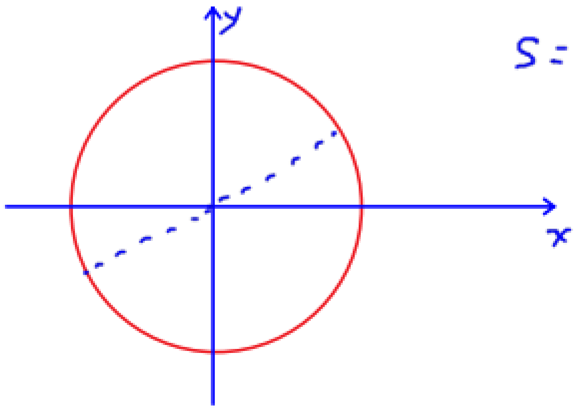
S/. $\tan\theta = 1 \iff \dfrac{\operatorname{sen}\theta}{\cos\theta} = 1 \iff \operatorname{sen}\theta = \cos\theta$



$$S = \left\{ \frac{\pi}{4} + n\pi \ , \ n = 0, 1, 2, 3, \dots \right\}$$

Ej: Resuelva $\tan\theta = \frac{1}{2}$ $\theta \in \mathbb{R}$

s/. $\tan\theta = \frac{1}{2} \Rightarrow \theta = \tan^{-1}\left(\frac{1}{2}\right) = 0.46$ rad.



$S = \{0.46 + n\pi,\ n = 0, 1, 2, \dots\}$

NOTA: Resolver $\cot\theta = \frac{1}{2}$ es equivalente a resolver $\tan\theta = 2$

Paréntesis. Dada $f(x) = 3x + \sqrt{x} - 1$. Calcule $f^{-1}(3)$

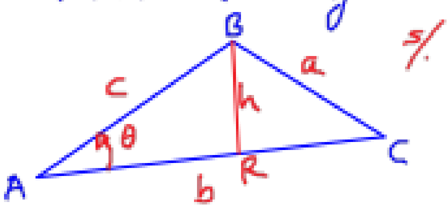
s/. Equivale a resolver $f(x) = 3$.

$3x + \sqrt{x} - 1 = 3 \iff \sqrt{x} = 4 - 3x \iff$

$x = (4-3x)^2 \iff x = 16 - 24x + 9x^2 \iff$

$9x^2 - 25x + 16 = 0$ ...

Ej: Calcule el área del triángulo ABC en función de b, c y θ



s/. h es una altura del triángulo; b es base

El ΔABR es rectángulo, por tanto $\operatorname{sen}\theta = \frac{h}{c} \Rightarrow h = c \cdot \operatorname{sen}\theta$

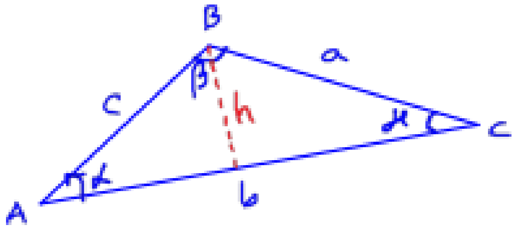
El área del ΔABC es:

$$\boxed{A = \frac{b \cdot h}{2} = \frac{1}{2} \cdot b \cdot c \cdot \operatorname{sen}\theta}$$

Trigonometría del triángulo

Ley de senos

α : alfa
β : beta
γ : gamma



$$\operatorname{sen}\alpha = \frac{h}{c} \qquad \operatorname{sen}\gamma = \frac{h}{a}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$c \cdot \operatorname{sen}\alpha = h \qquad a \cdot \operatorname{sen}\gamma = h$$

$$c \cdot \operatorname{sen}\alpha = a \operatorname{sen}\gamma \implies \frac{\operatorname{sen}\alpha}{a} = \frac{\operatorname{sen}\gamma}{c}$$

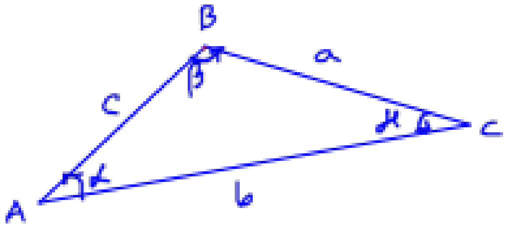
Si se tomara otra altura se podría probar que

$\dfrac{\operatorname{sen}\gamma}{c} = \dfrac{\operatorname{sen}\beta}{b}$. En conclusión :

$$\boxed{\frac{\operatorname{sen}\alpha}{a} = \frac{\operatorname{sen}\gamma}{c} = \frac{\operatorname{sen}\beta}{b}}$$

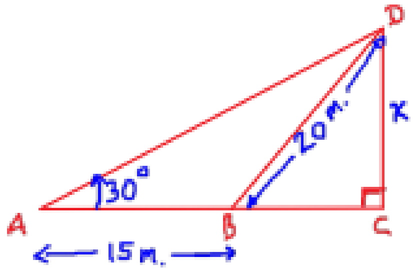
Ley de senos

Ley de cosenos



$$a^2 = b^2+c^2-2bc\cdot\cos\alpha$$
$$b^2 = a^2+c^2-2ac\cos\beta$$
$$c^2 = a^2+b^2-2ab\cos\gamma$$

Ej:



¿Con la información dada es posible calcular el valor de x?

S/ Sea $|\overline{BC}| = y$:

$$x^2+y^2=20^2 \quad\Downarrow\quad y=\sqrt{400-x^2} \qquad \tan 30^\circ = \frac{x}{y+15}$$

$$\Rightarrow \tan 30^\circ = \frac{x}{\sqrt{400-x^2}+15} \iff \left(\sqrt{400-x^2}+15\right)\tan 30^\circ = x$$

$$\iff \frac{1}{\sqrt{3}}\sqrt{400-x^2} + \frac{15}{\sqrt{3}} = x \iff \frac{1}{\sqrt{3}}\sqrt{400-x^2} = x - \frac{15}{\sqrt{3}}$$

$$\frac{1}{3}(400-x^2) = \left(x-\frac{15}{\sqrt{3}}\right)^2 = x^2 - \frac{2\,15}{\sqrt{3}}x - \frac{15^2}{3} \iff$$

. . .

¿Se podrá resolver de otra forma?